5-4 Complex Numbers

Main Ideas

- Find square roots and perform operations with pure imaginary numbers.
- Perform operations with complex numbers.

New Vocabulary

square root
imaginary unit
pure imaginary number
Square Root Property
complex number
complex conjugates

GET READY for the Lesson

Concepts in Motion
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Consider $2x^2 + 2 = 0$. One step in the solution of this equation is $x^2 = -1$. Since there is no real number that has a square of -1, there are no real solutions. French mathematician René Descartes (1596–1650) proposed that a number i be defined such that $i^2 = -1$.

Square Roots and Pure Imaginary Numbers A **square root** of a number n is a number with a square of n. For example, 7 is a square root of 49 because $7^2 = 49$. Since $(-7)^2 = 49$, -7 is also a square root of 49. Two properties will help you simplify expressions that contain square roots.

KEY CONCEPT — Product and Quotient Properties of Square Roots

Words For nonnegative real numbers a and b,

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, and

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, $b \neq 0$.

Examples $\sqrt{3 \cdot 2} = \sqrt{3} \cdot \sqrt{2}$

$\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}}$

Simplified square root expressions do not have radicals in the denominator, and any number remaining under the square root has no perfect square factor other than 1.

EXAMPLE 1 Properties of Square Roots

Simplify.

a. $\sqrt{50}$

$$\sqrt{50} = \sqrt{25 \cdot 2}$$
$$= \sqrt{25} \cdot \sqrt{2}$$
$$= 5\sqrt{2}$$

b. $\sqrt{\frac{11}{49}}$

$$\sqrt{\frac{11}{49}} = \frac{\sqrt{11}}{\sqrt{49}}$$
$$= \frac{\sqrt{11}}{7}$$

CHECK Your Progress

1A. $\sqrt{45}$

1B. $\sqrt{\frac{32}{81}}$

Since i is defined to have the property that $i^2 = -1$, the number i is the principal square root of -1; that is, $i = \sqrt{(-1)}$. i is called the **imaginary unit**. Numbers of the form $3i$, $-5i$, and $i\sqrt{2}$ are called **pure imaginary numbers**. Pure imaginary numbers are square roots of negative real numbers. For any positive real number b, $\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$ or bi.

Reading Math

Imaginary Unit i is usually written before radical symbols to make it clear that it is not under the radical.

EXAMPLE 2 Square Roots of Negative Numbers

Simplify.

a. $\sqrt{-18}$

$\sqrt{-18} = \sqrt{-1 \cdot 3^2 \cdot 2}$

$= \sqrt{-1} \cdot \sqrt{3^2} \cdot \sqrt{2}$

$= i \cdot 3 \cdot \sqrt{2}$ or $3i\sqrt{2}$

b. $\sqrt{-125x^5}$

$\sqrt{-125x^5} = \sqrt{-1 \cdot 5^2 \cdot x^4 \cdot 5x}$

$= \sqrt{-1} \cdot \sqrt{5^2} \cdot \sqrt{x^4} \cdot \sqrt{5x}$

$= i \cdot 5 \cdot x^2 \cdot \sqrt{5x}$ or $5ix^2\sqrt{5x}$

CHECK Your Progress

2A. $\sqrt{-27}$

2B. $\sqrt{-216y^4}$

The Commutative and Associative Properties of Multiplication hold true for pure imaginary numbers.

EXAMPLE 3 Products of Pure Imaginary Numbers

Simplify.

a. $-2i \cdot 7i$

$-2i \cdot 7i = -14i^2$

$= -14(-1)$ $\quad i^2 = -1$

$= 14$

b. $\sqrt{-10} \cdot \sqrt{-15}$

$\sqrt{-10} \cdot \sqrt{-15} = i\sqrt{10} \cdot i\sqrt{15}$

$= i^2\sqrt{150}$

$= -1 \cdot \sqrt{25} \cdot \sqrt{6}$

$= -5\sqrt{6}$

c. i^{45}

$i^{45} = i \cdot i^{44}$ — Multiplying powers

$= i \cdot (i^2)^{22}$ — Power of a Power

$= i \cdot (-1)^{22}$ — $i^2 = -1$

$= i \cdot 1$ or i — $(-1)^{22} = 1$

CHECK Your Progress

3A. $3i \cdot 4i$

3B. $\sqrt{-20} \cdot \sqrt{-12}$

3C. i^{31}

You can solve some quadratic equations by using the **Square Root Property**.

Reading Math

Plus or Minus $\pm\sqrt{n}$ is read *plus or minus the square root of n.*

KEY CONCEPT *Square Root Property*

For any real number n, if $x^2 = n$, then $x = \pm\sqrt{n}$.

EXAMPLE 4 Equation with Pure Imaginary Solutions

Solve $3x^2 + 48 = 0$.

$3x^2 + 48 = 0$ — Original equation

$3x^2 = -48$ — Subtract 48 from each side.

$x^2 = -16$ — Divide each side by 3.

$x = \pm\sqrt{-16}$ — Square Root Property

$x = \pm 4i$ — $\sqrt{-16} = \sqrt{16} \cdot \sqrt{-1}$

CHECK Your Progress Solve each equation.

4A. $4x^2 + 100 = 0$ **4B.** $x^2 + 4 = 0$

Operations with Complex Numbers

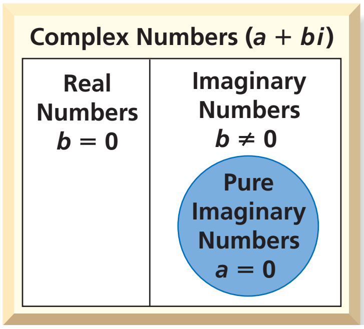
Consider $5 + 2i$. Since 5 is a real number and $2i$ is a pure imaginary number, the terms are not like terms and cannot be combined. This type of expression is called a **complex number**.

KEY CONCEPT *Complex Numbers*

Words	A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. a is called the real part, and b is called the imaginary part.
Examples	$7 + 4i$ and $2 - 6i = 2 + (-6)i$

The Venn diagram shows the complex numbers.

- If $b = 0$, the complex number is a real number.
- If $b \neq 0$, the complex number is imaginary.
- If $a = 0$, the complex number is a pure imaginary number.



Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Reading Math

Complex Numbers The form $a + bi$ is sometimes called the *standard form* of a complex number.

EXAMPLE 5 Equate Complex Numbers

Find the values of x and y that make the equation $2x - 3 + (y - 4)i = 3 + 2i$ true.

Set the real parts equal to each other and the imaginary parts equal to each other.

$2x - 3 = 3$ Real parts

$2x = 6$ Add 3 to each side.

$x = 3$ Divide each side by 2.

$y - 4 = 2$ Imaginary parts

$y = 6$ Add 4 to each side.

CHECK Your Progress

5. Find the values of x and y that make the equation $5x + 1 + (3 + 2y)i = 2x - 2 + (y - 6)i$ true.

To add or subtract complex numbers, combine like terms. That is, combine the real parts and combine the imaginary parts.

EXAMPLE 6 Add and Subtract Complex Numbers

Simplify.

a. $(6 - 4i) + (1 + 3i)$

$(6 - 4i) + (1 + 3i) = (6 + 1) + (-4 + 3)i$ Commutative and Associative Properties

$= 7 - i$ Simplify.

b. $(3 - 2i) - (5 - 4i)$

$(3 - 2i) - (5 - 4i) = (3 - 5) + [-2 - (-4)]i$ Commutative and Associative Properties

$= -2 + 2i$ Simplify.

CHECK Your Progress

6A. $(-2 + 5i) + (1 - 7i)$ **6B.** $(4 + 6i) - (-1 + 2i)$

Study Tip

Complex Numbers

While all real numbers are also complex, the term *Complex Numbers* usually refers to a number that is not real.

One difference between real and complex numbers is that complex numbers cannot be represented by lines on a coordinate plane. However, complex numbers can be graphed on a *complex plane.* A complex plane is similar to a coordinate plane, except that the horizontal axis represents the real part a of the complex number, and the vertical axis represents the imaginary part b of the complex number.

You can also use a complex plane to model the addition of complex numbers.

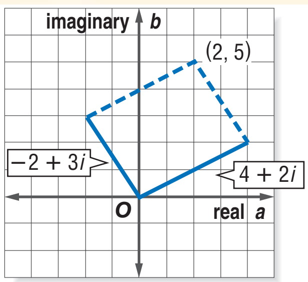
ALGEBRA LAB

Adding Complex Numbers Graphically

Use a complex plane to find $(4 + 2i) + (-2 + 3i)$.

- Graph $4 + 2i$ by drawing a segment from the origin to (4, 2) on the complex plane.
- Graph $-2 + 3i$ by drawing a segment from the origin to (−2 , 3) on the complex plane.
- Given three vertices of a parallelogram, complete the parallelogram.
- The fourth vertex at (2, 5) represents the complex number $2 + 5i$.

So, $(4 + 2i) + (-2 + 3i) = 2 + 5i$.



MODEL AND ANALYZE

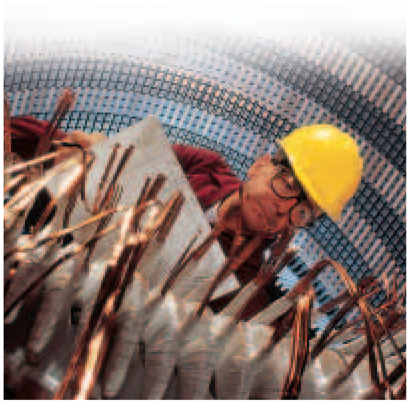
1. Model $(-3 + 2i) + (4 - i)$ on a complex plane.

2. Describe how you could model the difference $(-3 + 2i) - (4 - i)$ on a complex plane.

Complex numbers are used with electricity. In a circuit with alternating current, the voltage, current, and impedance, or hindrance to current, can be represented by complex numbers. To multiply these numbers, use the FOIL method.

Study Tip

Electrical engineers use ***j*** as the imaginary unit to avoid confusion with the ***I*** for current.



Real-World Career

Electrical Engineer

The chips and circuits in computers are designed by electrical engineers.

Math Online

For more information, go to algebra2.com.

Real-World EXAMPLE

7 **ELECTRICITY** **In an AC circuit, the voltage *E*, current *I*, and impedance *Z* are related by the formula $E = I \cdot Z$. Find the voltage in a circuit with current $1 + 3j$ amps and impedance $7 - 5j$ ohms.**

$E = I \cdot Z$	Electricity formula
$= (1 + 3j) \cdot (7 - 5j)$	$I = 1 + 3j, Z = 7 - 5j$
$= 1(7) + 1(-5j) + (3j)7 + 3j(-5j)$	FOIL
$= 7 - 5j + 21j - 15j^2$	Multiply.
$= 7 + 16j - 15(-1)$	$j^2 = -1$
$= 22 + 16j$	Add.

The voltage is $22 + 16j$ volts.

CHECK Your Progress

7. Find the voltage in a circuit with current $2 - 4j$ amps and impedance $3 - 2j$ ohms.

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Two complex numbers of the form $a + bi$ and $a - bi$ are called **complex conjugates**. The product of complex conjugates is always a real number. You can use this fact to simplify the quotient of two complex numbers.

EXAMPLE Divide Complex Numbers

8 **Simplify.**

a. $\frac{3i}{2 + 4i}$

$\frac{3i}{2 + 4i} = \frac{3i}{2 + 4i} \cdot \frac{2 - 4i}{2 - 4i}$	$2 + 4i$ and $2 + 4i$ are conjugates.
$= \frac{6i - 12i^2}{4 - 16i^2}$	Multiply.
$= \frac{6i + 12}{20}$	$i^2 = -1$
$= \frac{3}{5} + \frac{3}{10}i$	Standard form

b. $\frac{5 + i}{2i}$

$\frac{5 + i}{2i} = \frac{5 + i}{2i} \cdot \frac{i}{i}$	Why multiply by $\frac{i}{i}$ instead of $\frac{-2i}{-2i}$?
$= \frac{5i + i^2}{2i^2}$	Multiply.
$= \frac{5i - 1}{-2}$	$i^2 = -1$
$= \frac{1}{2} - \frac{5}{2}i$	Standard form

CHECK Your Progress

8A. $\frac{-2i}{3 + 5i}$ **8B.** $\frac{2 + i}{1 - i}$

CHECK Your Understanding

Examples 1–3 (pp. 259–260)

Simplify.

1. $\sqrt{56}$
2. $\sqrt{80}$
3. $\sqrt{\frac{48}{49}}$
4. $\sqrt{\frac{120}{9}}$
5. $\sqrt{-36}$
6. $\sqrt{-50x^2y^2}$
7. $(6i)(-2i)$
8. $5\sqrt{-24} \cdot 3\sqrt{-18}$
9. i^{29}
10. i^{80}

Example 4 (p. 260)

Solve each equation.

11. $2x^2 + 18 = 0$
12. $-5x^2 - 25 = 0$

Example 5 (p. 261)

Find the values of m and n that make each equation true.

13. $2m + (3n + 1)i = 6 - 8i$
14. $(2n - 5) + (-m - 2)i = 3 - 7i$

Example 6 (p. 262)

15. **ELECTRICITY** The current in one part of a series circuit is $4 - j$ amps. The current in another part of the circuit is $6 + 4j$ amps. Add these complex numbers to find the total current in the circuit.

Examples 7, 8 (p. 263)

Simplify.

16. $(-2 + 7i) + (-4 - 5i)$
17. $(8 + 6i) - (2 + 3i)$
18. $(3 - 5i)(4 + 6i)$
19. $(1 + 2i)(-1 + 4i)$
20. $\frac{2 - i}{5 + 2i}$
21. $\frac{3 + i}{1 + 4i}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
22–25	1
26–29	2
30–33	3
34–37	6
38, 39, 50	7
40, 41, 51	8
42–45	4
46–49	5

Simplify.

22. $\sqrt{125}$
23. $\sqrt{147}$
24. $\sqrt{\frac{192}{121}}$
25. $\sqrt{\frac{350}{81}}$
26. $\sqrt{-144}$
27. $\sqrt{-81}$
28. $\sqrt{-64x^4}$
29. $\sqrt{-100a^4b^2}$
30. $(-2i)(-6i)(4i)$
31. $3i(-5i)^2$
32. i^{13}
33. i^{24}
34. $(5 - 2i) + (4 + 4i)$
35. $(-2 + i) + (-1 - i)$
36. $(15 + 3i) - (9 - 3i)$
37. $(3 - 4i) - (1 - 4i)$
38. $(3 + 4i)(3 - 4i)$
39. $(1 - 4i)(2 + i)$
40. $\frac{4i}{3 + i}$
41. $\frac{4}{5 + 3i}$

Solve each equation.

42. $5x^2 + 5 = 0$
43. $4x^2 + 64 = 0$
44. $2x^2 + 12 = 0$
45. $6x^2 + 72 = 0$

Find the values of m and n that make each equation true.

46. $8 + 15i = 2m + 3ni$
47. $(m + 1) + 3ni = 5 - 9i$
48. $(2m + 5) + (1 - n)i = -2 + 4i$
49. $(4 + n) + (3m - 7)i = 8 - 2i$

ELECTRICITY For Exercises 50 and 51, use the formula $E = I \cdot Z$.

50. The current in a circuit is $2 + 5j$ amps, and the impedance is $4 - j$ ohms. What is the voltage?

51. The voltage in a circuit is $14 - 8j$ volts, and the impedance is $2 - 3j$ ohms. What is the current?

52. Find the sum of $ix^2 - (2 + 3i)x + 2$ and $4x^2 + (5 + 2i)x - 4i$.

53. Simplify $[(3 + i)x^2 - ix + 4 + i] - [(-2 + 3i)x^2 + (1 - 2i)x - 3]$.

Simplify.

54. $\sqrt{-13} \cdot \sqrt{-26}$ **55.** $(4i)\left(\frac{1}{2}i\right)^2(-2i)^2$ **56.** i^{38}

57. $(3 - 5i) + (3 + 5i)$ **58.** $(7 - 4i) - (3 + i)$ **59.** $(-3 - i)(2 - 2i)$

60. $\dfrac{(10 + i)^2}{4 - i}$ **61.** $\dfrac{2 - i}{3 - 4i}$

62. $(-5 + 2i)(6 - i)(4 + 3i)$ **63.** $(2 + i)(1 + 2i)(3 - 4i)$

64. $\dfrac{5 - i\sqrt{3}}{5 + i\sqrt{3}}$ **65.** $\dfrac{1 - i\sqrt{2}}{1 + i\sqrt{2}}$

Solve each equation, and locate the complex solutions in the complex plane.

66. $-3x^2 - 9 = 0$ **67.** $-2x^2 - 80 = 0$

68. $\frac{2}{3}x^2 + 30 = 0$ **69.** $\frac{4}{5}x^2 + 1 = 0$

EXTRA PRACTICE
See pages 900, 930.
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Find the values of m and n that make each equation true.

70. $(m + 2n) + (2m - n)i = 5 + 5i$ **71.** $(2m - 3n)i + (m + 4n) = 13 + 7i$

72. ELECTRICITY The impedance in one part of a series circuit is $3 + 4j$ ohms, and the impedance in another part of the circuit is $2 - 6j$. Add these complex numbers to find the total impedance in the circuit.

H.O.T. Problems

73. OPEN ENDED Write two complex numbers with a product of 10.

74. CHALLENGE Copy and complete the table. Explain how to use the exponent to determine the simplified form of any power of i.

Power of i	Simplified Expression
i^6	?
i^7	?
i^8	?
i^9	?
i^{10}	?
i^{11}	?
i^{12}	?
i^{13}	?

75. Which One Doesn't Belong? Identify the expression that does not belong with the other three. Explain your reasoning.

$(3i)^2$ $(2i)(3i)(4i)$ $(6 + 2i) - (4 + 2i)$ $(2i)^4$

76. REASONING Determine if each statement is *true* or *false*. If false, find a counterexample.

a. Every real number is a complex number.

b. Every imaginary number is a complex number.

77. *Writing in Math* Use the information on page 261 to explain how complex numbers are related to quadratic equations. Explain how the a and c must be related if the equation $ax^2 + c = 0$ has complex solutions and give the solutions of the equation $2x^2 + 2 = 0$.

STANDARDIZED TEST PRACTICE

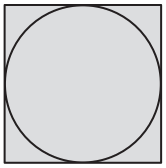
78. **ACT/SAT** The area of the square is 16 square units. What is the area of the circle?

A 2π units^2

B 12 units^2

C 4π units^2

D 16π units^2



79. If $i^2 = -1$, then what is the value of i^{71}?

F -1

G 0

H $-i$

J i

Spiral Review

Write a quadratic equation with the given root(s). Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers. (Lesson 5-3)

80. $-3, 9$

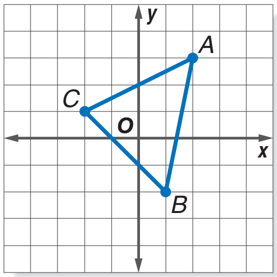
81. $-\frac{1}{3}, -\frac{3}{4}$

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located. (Lesson 5-2)

82. $3x^2 = 4 - 8x$

83. $2x^2 + 11x = -12$

Triangle *ABC* is reflected over the *x*-axis. (Lesson 4-4)



84. Write a vertex matrix for the triangle.

85. Write the reflection matrix.

86. Write the vertex matrix for $\triangle A'B'C'$.

87. Graph $\triangle A'B'C'$.

88. **FURNITURE** A new sofa, love seat, and coffee table cost \$2050. The sofa costs twice as much as the love seat. The sofa and the coffee table together cost \$1450. How much does each piece of furniture cost? (Lesson 3-5)

89. **DECORATION** Samantha is going to use more than 75 but less than 100 bricks to make a patio off her back porch. If each brick costs \$2.75, write and solve a compound inequality to determine the amount she will spend on bricks. (Lesson 1-6)

GET READY for the Next Lesson

Determine whether each polynomial is a perfect square trinomial. (Lesson 5-3)

90. $x^2 - 10x + 16$

91. $x^2 + 18x + 81$

92. $x^2 - 9$

93. $x^2 - 12x - 36$

94. $x^2 - x + \frac{1}{4}$

95. $2x^2 - 15x + 25$